

Nanospintronics meets relativistic quantum physics: Ubiquity of Zitterbewegung effects

U. Zülicke^{a,1}, R. Winkler^b and J. Bolte^c

^a*Institute of Fundamental Sciences and MacDiarmid Institute for Advanced Materials and Nanotechnology, Massey University, Private Bag 11 222, Palmerston North, New Zealand*

^b*Department of Physics, Northern Illinois University, DeKalb, IL 60115*

^c*Institut für Theoretische Physik, Universität Ulm, Albert-Einstein-Allee 11, D-89069 Ulm, Germany*

Abstract

We present a unified description of *zitterbewegung*-like phenomena for electron and hole systems showing Rashba spin splitting as well as for electrons in single-layer and bilayer graphene. The former class of systems can be interpreted as “nonrelativistic” whereas the latter are often called “ultrarelativistic” so that our unified description indicates an interesting connection between these two opposite limits.

Key words: Zitterbewegung, Dirac equation, multiband Hamiltonians, nonrelativistic limit, ultrarelativistic limit
PACS: 73.21.-b, 71.70.Ej, 03.65.Pm

Zitterbewegung (German for trembling motion) is a highly oscillatory component in the orbital motion of free Dirac particles [1] which is commonly considered to be a quirk of the very successful Dirac theory of relativistic electron dynamics. Its physical significance has been discussed controversially over the years. Suggestions [2] for a possibly observable *zitterbewegung*-like dynamics of band electrons in solids have lifted this discussion onto a new level. The Rashba model [3] studied in Ref. [2] is equivalent to the Pauli equation, the latter being the nonrelativistic limit of the Dirac equation. Recently, graphene has become the subject of intensive research [4]. Its electronic structure bears remarkable analogies with the ultrarelativistic limit of the Dirac equation [5, 6]. *Zitterbewegung*-like phenomena in graphene were recently studied in Refs. [7, 8]. Here we present a unified description for *zitterbewegung*-like phenomena in the “nonrelativistic” Rashba and the “ultrarelativistic” graphene

¹ Corresponding author. E-mail: u.zuelicke@massey.ac.nz

cases based on Refs. [9, 10] which indicates an interesting connection between these opposite limiting cases.

We analyze the generalized 2D Rashba Hamiltonian

$$H_R^{(n)} = i\beta (p_-^n \sigma_+ - p_+^n \sigma_-) = \beta \begin{pmatrix} 0 & i p_-^n \\ -i p_+^n & 0 \end{pmatrix}, \quad (1)$$

where β is a system-dependent prefactor, $p_{\pm} \equiv p_x \pm i p_y$, $\sigma_{\pm} \equiv (\sigma_x \pm i \sigma_y)/2$, and σ_x, σ_y denote Pauli spin matrices. The Hamiltonian $H_R^{(1)}$ is the usual Rashba Hamiltonian [3]; for $n = 3$ we get the Hamiltonian that describes spin splitting in heavy-hole systems [11]. We omit here the spin-independent kinetic-energy term which, in our context, results in a trivial correction [10]. The Hamiltonian $H_R^{(1)}$ is equivalent to the well-known Pauli spin-orbit coupling [12]. The Pauli equation represents the nonrelativistic limit to the Dirac equation.

Quasi-free electrons in graphene can be described by the effective 2D Hamiltonians

$$H_g^{(n)} = \beta (p_-^n \sigma_+ + p_+^n \sigma_-) = \beta \begin{pmatrix} 0 & p_-^n \\ p_+^n & 0 \end{pmatrix}. \quad (2)$$

The Hamiltonian $H_g^{(1)}$ describes electrons in single-layer graphene [5, 6]; and $H_g^{(2)}$ is appropriate for bilayer graphene [13]. Moreover, the Hamiltonian $H_g^{(1)}$ is equivalent to the ultrarelativistic limit of the 2D Dirac Hamiltonian. It is remarkable that all our subsequent results hold for both classes of Hamiltonians, $H_R^{(n)}$ and $H_g^{(n)}$, which indicates an interesting connection between the nonrelativistic and the ultrarelativistic limit of the Dirac equation. From a formal point of view, this result reflects the fact that $H_R^{(n)}$ and $H_g^{(n)}$ are related to each other by a simple unitary transformation. In the following, the Hamiltonian is thus simply denoted by H .

Following Refs. [9, 10], we express the Heisenberg velocity operator for H in the form $\mathbf{v}(t) = \bar{\mathbf{v}}(t) + \tilde{\mathbf{v}}(t)$, where the mean part is $\bar{\mathbf{v}}(t) = \partial H / \partial \mathbf{p} - \mathbf{F}$, and the oscillating part describing the *zitterbewegung*-like motion reads $\tilde{\mathbf{v}}(t) = \mathbf{F} e^{-i\hat{\omega}(\mathbf{p})t}$. Here the amplitude operator \mathbf{F} and the frequency operator $\hat{\omega}(\mathbf{p})$ are given by

$$\mathbf{F} = \frac{\partial H}{\partial \mathbf{p}} - \frac{n\mathbf{p}H}{p^2} \equiv \sigma_z \mathbf{e}_z \times \mathbf{p} \frac{nH}{ip^2}, \quad \hat{\omega}(\mathbf{p}) = \frac{2H}{\hbar}, \quad (3)$$

where \mathbf{e}_z is a unit vector perpendicular to the 2D plane. Note that \mathbf{F} depends explicitly on the index n . The Heisenberg position operator $\mathbf{r}(t)$ can be expressed in a similar way as $\mathbf{r}(t) = \bar{\mathbf{r}}(t) + \tilde{\mathbf{r}}(t)$, where $\bar{\mathbf{r}}(t) = \mathbf{r} + \bar{\mathbf{v}}t + \mathbf{F} \frac{1}{i\hat{\omega}(\mathbf{p})}$ and $\tilde{\mathbf{r}}(t) = -\mathbf{F} \frac{e^{-i\hat{\omega}(\mathbf{p})t}}{i\hat{\omega}(\mathbf{p})}$. For the particular case of the Rashba Hamiltonian

$H = H_r^{(1)}$, our general expression for $\mathbf{r}(t)$ agrees with Refs. [2, 10]. For single-layer graphene, $H = H_g^{(1)}$, it reproduces the results in Ref. [7], and for bilayer graphene, $H = H_g^{(2)}$, it agrees with Ref. [8].

Explicit evaluation shows that $\mathbf{v}(t)$ oscillates with the frequency $\omega = 2\beta p^n/\hbar$. The oscillations become arbitrarily slow for $p \rightarrow 0$. We obtain an estimate for the amplitude of the oscillations from $\tilde{r}^2(t) = (n/2)^2 \lambda_B^2$, where $\lambda_B = \hbar/p$ is the de Broglie wave length. Note that this expression is independent of the prefactor β and it diverges in the limit $p \rightarrow 0$. In a similar way, we obtain an estimate for the magnitude of the velocity of the oscillatory motion from

$$\tilde{v}^2(t) = \left(n\beta p^{n-1}\right)^2 = \left(\frac{n}{2}\omega\lambda_B\right)^2. \quad (4)$$

The components of $\mathbf{v}(t)$ do not commute and we have

$$[v_x(t), v_y(t)] = 2i\left(n\beta p^{n-1}\right)^2 \sigma_z e^{-i\hat{\omega}(\mathbf{p})t}, \quad (5a)$$

which implies the uncertainty relation

$$\Delta v_x \Delta v_y \geq \left(n\beta p^{n-1}\right)^2. \quad (5b)$$

Thus both the magnitude of the oscillations of $\mathbf{v}(t)$ and the minimum uncertainty for a simultaneous measurement of v_x and v_y are given by the same combination of parameters [10].

Zitterbewegung-like phenomena are manifested also by oscillations of the z -components of orbital angular momentum $L_z(t) = [\mathbf{r}(t) \times \mathbf{p}]_z$ and (pseudo)spin $S_z(t)$. We get

$$L_z(t) = L_z + \frac{n\hbar\sigma_z}{2} \left(1 - e^{-i\hat{\omega}(\mathbf{p})t}\right), \quad (6a)$$

$$S_z(t) = \frac{n\hbar\sigma_z}{2} e^{-i\hat{\omega}(\mathbf{p})t}. \quad (6b)$$

The appearance of the index n in Eq. (6b) reflects the fact that the 2D Hamiltonian $H^{(n)}$ describes effective particles with pseudospin z -component $\pm n/2$. Electrons in the linear Rashba model as well as electrons in single-layer graphene have pseudospin $\pm 1/2$, electrons in bilayer graphene are characterized by a pseudospin ± 1 , and heavy holes described by the cubic Rashba model have pseudospin $\pm 3/2$. We note also that for $H = H_r^{(1)}$, Eq. (6b) represents the well-known spin precession in the effective magnetic field of the Rashba term, which has been observed experimentally [14]. The time dependence in Eqs. (6a) and (6b) reflects the fact that L_z and S_z are not individually conserved. However, as expected for models of free quasiparticles, the total angular momentum J_z is conserved

$$J_z(t) = J_z = L_z + S_z, \quad (6c)$$

which reflects the fact that $[J_z, H] = 0$.

In conclusion, we have presented a unified description of *zitterbewegung*-like phenomena for electron and hole systems showing Rashba spin splitting as well as for electrons in single-layer and bilayer graphene. The former class of systems can be interpreted as “nonrelativistic” whereas the latter are often called “ultrarelativistic” so that our unified description indicates an interesting connection between these two opposite limits.

The authors appreciate stimulating discussions with B. Trauzettel. UZ is supported by the Marsden Fund Council from Government funding, administered by the Royal Society of New Zealand.

References

- [1] E. Schrödinger, Sitzber. Preuss. Akad. Wiss., Phys.-Math. Kl. **24**, 418 (1930).
- [2] J. Schliemann, D. Loss, and R. M. Westervelt, Phys. Rev. Lett. **94**, 206801 (2005).
- [3] E. I. Rashba, Sov. Phys.–Solid State **2**, 1109 (1960).
- [4] A. K. Geim and K. S. Novoselov, Nature Mat. **6**, 183 (2007).
- [5] P. R. Wallace, Phys. Rev. **71**, 622 (1947).
- [6] D. P. DiVincenzo and E. J. Mele, Phys. Rev. B **29**, 1685 (1984).
- [7] M. Katsnelson, Eur. Phys. J. B **51**, 157 (2006).
- [8] J. Cserti and G. Dávid, Phys. Rev. B **74**, 172305 (2006).
- [9] B. Thaller, *The Dirac Equation* (Springer, Berlin, 1992).
- [10] R. Winkler, U. Zülicke, and J. Bolte, Phys. Rev. B **75**, 205314 (2007).
- [11] R. Winkler, *Spin-Orbit Coupling Effects in Two-Dimensional Electron and Hole Systems* (Springer, Berlin, 2003).
- [12] J. J. Sakurai, *Advanced Quantum Mechanics* (Addison-Wesley, Reading, MA, 1967).
- [13] E. McCann and V. I. Fal’ko, Phys. Rev. Lett. **96**, 086805 (2006).
- [14] S. A. Crooker and D. L. Smith, Phys. Rev. Lett. **94**, 236601 (2005).